On the geometric structure of non-hyperbolic attractors

1988 J. Phys. A: Math. Gen. 21 L763

(http://iopscience.iop.org/0305-4470/21/15/004)

J. Phys. A: Math. Gen. 21 (1988) L763–L769. Printed in the UK

LETTER TO THE EDITOR

On the geometric structure of non-hyperbolic attractors

A Politi†, R Badii‡ and P Grassberger§

† Istituto Nazionale di Ottica, Largo Enrico Fermi 6, I-50125 Firenze, Italy
‡ Department of Chemical Physics, The Weizmann Institute of Science, IL-76100, Rehovot, Israel
§ Physics Department, University of Wuppertal, D-5600 Wuppertal 1, Federal Republic of Germany

Received 7 April 1988

Abstract. We discuss the $f(\alpha)$ spectrum of non-hyperbolic attractors of the Hénon type. We elucidate the origin of the 'phase transition' found in a previous paper, and give a lower bound to the spectrum in the non-hyperbolic 'phase' where Kaplan–Yorke-type formulae no longer hold. Our results disagree with other recent attempts. Numerical simulations for the Hénon map agree with analytical estimates.

Very recently, much progress has been done towards the understanding of the geometric structure of strange attractors. In the case of hyperbolic systems, i.e. of sets whose tangent space is a continuous invariant decomposition of the stable and unstable eigenspaces, the generalised dimensions $D(q)$ (Renyi 1970, Grassberger 1983, Hentschel and Procaccia 1983) have been shown to be related (Grassberger 1984, Badii and Politi 1987) to the effective (finite-time) Lyapunov exponents (Fujisaka 1983, Grassberger and Procaccia 1983, Benzi *et al* 1985). The latter also yield (Grassberger *et al* 1987, Mori *et al* 1987a, b) the whole spectrum $f(\alpha)$ (Parisi (appendix to Frisch (1984)), Halsey *et al* 1986) of pointwise dimensions α. These results have been derived under the assumption that a hyperbolic attractor is, locally, a cartesian product of several continua and one Cantor set. On the other hand, very little is known about the more general class of non-hyperbolic systems for which a thermodynamic formalism has not yet been developed. However, very recently, a first heuristic attempt to extend the statistical mechanics approach to such systems has led to the conjecture (Grassberger *et al* 1987) that the above relations still hold for α larger than some critical value α_c. For smaller α values, the effect of homoclinic tangencies prevails. The factorisation hypothesis (continua × Cantor set) breaks down there and must be substituted by more refined considerations. Here, we present such an approach, which allows us to analyse the low-dimension tail of the spectrum $f(\alpha)$, by studying the natural measure around the homoclinic tangencies. For two-dimensional maps in the high-dissipation limit, we recover the well known results for the logistic map (triangular $f(\alpha)$ spectrum) (Grassberger *et al* 1987). We first recall the definitions of generalised (Renyi) dimensions $D(q)$ and the $f(\alpha)$ spectrum. The former can be defined through (Grassberger 1985, Halsey *et al* 1986)

$$\lim_{\varepsilon_i\to 0}\begin{Bmatrix}\sup\\ \inf\end{Bmatrix}\sum_i \frac{p_i^q}{\varepsilon_i^{\tau(q)}} = \mathrm{O}(1) \qquad \text{for} \qquad \begin{cases} q>1\\ q<1\end{cases} \qquad \tau(q)=(q-1)D(q) \tag{1}$$

where the attractor has been covered by compact sets ('boxes') of size ε_i containing a mass p_i. The pointwise dimension $\alpha(x)$ is then introduced via the scaling relation $p_i \sim \varepsilon_i^{\alpha(x)}$ between mass and size, for the box centred at point x. Finally, the dimension spectrum $f(\alpha)$ is defined as the Legendre transform of $\tau(q)$:

$$f(\alpha) = q\alpha(q) - \tau(q) \qquad \text{with} \qquad \alpha = \mathrm{d}\tau(q)/\mathrm{d}q. \tag{2}$$

According to Parisi (appendix to Frisch (1984)), $f(\alpha)$ can be interpreted as the Hausdorff dimension of the set of points x with pointwise dimension $\alpha(x) = \alpha$. For hyperbolic repellers in one dimension, this was shown rigorously by Bohr and Rand (1987) and Collet *et al* (1987). We shall see in the following that this is no longer obvious for non-hyperbolic systems.

It has been shown that a relation between $D(q)$ and effective Lyapunov exponents exists for hyperbolic systems (Grassberger 1984, Badii and Politi 1987). For simplicity, we recall only the results for 2D maps with constant Jacobian $J \equiv \exp(-B)$. By indicating the partial dimensions along the expanding and contracting directions with $d_1(q) = 1$ and $d_2(q)$, respectively, the global dimension is $D(q) = 1 + d_2(q)$ and the following generalised volume conservation law (Grassberger 1984, Badii and Politi 1987) holds:

$$\lim_{t\to\infty} \langle \exp[(1-q)[\lambda_1(t,x) + d_2(q)\lambda_2(t,x)]t]\rangle = 1 \tag{3}$$

where $\lambda_1(t,x)$, $\lambda_2(t,x)$ are the effective Lyapunov exponents computed over a portion of orbit of time length t, starting from point x. By using $\lambda_1 + \lambda_2 = -B$, and defining the generalised Lyapunov exponent $\Lambda_1(q)$ as (Fujisaka 1983, Grassberger and Procaccia 1983, Benzi *et al* 1985)

$$\Lambda_1(q) = \lim_{t\to\infty} \frac{1}{(1-q)t} \ln\langle \exp[(1-q)\lambda_1(t,x)t]\rangle \tag{4}$$

one finds

$$\Lambda_1[q-(q-1)d_2(q)] = \frac{Bd_2(q)}{1-d_2(q)}. \tag{5}$$

In non-hyperbolic systems of the Hénon type, the value of $\Lambda_1(q)$ depends, for $q > 2$, on details of the definition, such as whether the averaging in (4) is done over open trajectories (which allow for negative Λ_1 values as well), or over periodic ones only (obviously yielding positive Λ_1 values only). These differences are related to non-analytic behaviour of $D(q)$ at the value $q = q_c$ which renders the argument of Λ_1 in (5) equal to 2 (Grassberger *et al* 1987):

$$[2 - D(q_c)](q_c - 1) = 1. \tag{6}$$

In such a point, the derivative of $D(q)$ is discontinuous, in analogy to a first-order phase transition: in fact, $\tilde{\alpha}_c \equiv \lim_{q\searrow q_c} \mathrm{d}\tau(q)/\mathrm{d}q < \alpha_c \equiv \lim_{q\nearrow q_c} \mathrm{d}\tau(q)/\mathrm{d}q$, and $f(\alpha)$ is linear for $\tilde{\alpha}_c \leqslant \alpha \leqslant \alpha_c$, with slope $\mathrm{d}f(\alpha)/\mathrm{d}\alpha = q_c$. For the usual parameters of the Hénon map ($a = 1.4$, $b = 0.3$), one finds numerically $\Lambda_1(2) = 0.29$ and, consequently, $q_c = 2.24$ and $D(q_c) = 1.19$ (from (5) and (6)). The shape of $f(\alpha)$ is given in figure 1. For $\alpha > \alpha_c$, $f(\alpha)$ is obtained directly from (5). The simplest way to complete the curve for $\alpha < \alpha_c$, consistent with the above results, would be to extrapolate it down to $f(\alpha) = 0$ with a straight line of slope q_c: this yields a lower bound $\alpha_{\min}^-$ to the minimum dimension

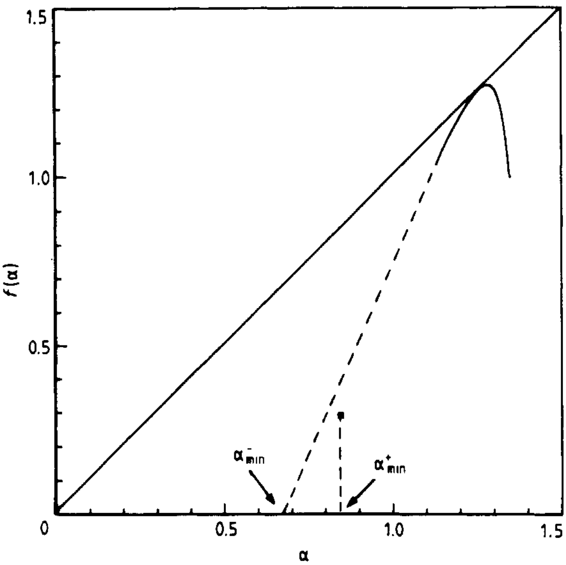

Figure 1. $f(\alpha)$ spectrum of the Hénon attractor. The broken line gives an upper bound to the low-dimension tail. The isolated point (■) represents an upper bound to the minimum α value and a lower bound to $f(\alpha)$.

α_{min} and an upper bound $f^+(\alpha)$ to the spectrum, defined as

$$\begin{aligned} \alpha^-_{min} &= \tilde{\alpha}_c = \frac{D(q_c)}{3 - D(q_c)} \qquad (=0.66 \quad \text{for} \quad a = 1.4, b = 0.3) \\ f^+(\alpha) &= (\alpha - \alpha^-_{min})q_c \qquad \text{for} \quad \alpha^-_{min} \leqslant \alpha \leqslant \alpha_c. \end{aligned} \tag{7}$$

The aim of the present letter is to investigate whether this bound is saturated. In particular, we determine lower and upper bounds to both $f(\alpha)$ and α_{min} by following a direct approach.

The main point is to recognise that the regions of the attractor which contribute most to the low-dimension tail of $f(\alpha)$ are the neighbourhoods of the homoclinic tangencies (figure 2). Notice that the image of a tangency point is again a tangency,

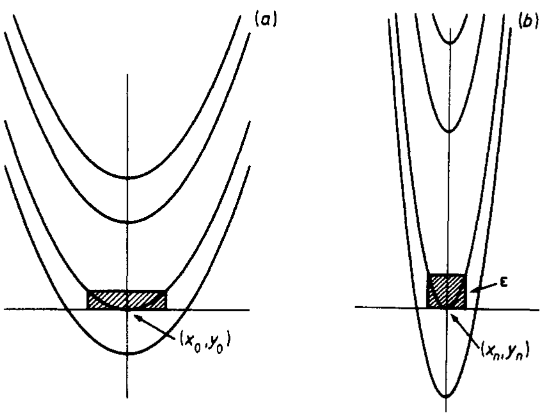

Figure 2. Sketch of the structure of the stable and unstable manifolds near a prominent homoclinic tangency (a) and around its nth iterate (b).

characterised by a larger curvature of the unstable manifold (and vice versa for the pre-images). Therefore, given a family of tangencies $\{x_n | x_{n+1} = F(x_n)\}$, we call 'prominent' tangency the point x_0 in which the sum of the curvatures of stable and unstable manifold is minimal. We then make the following assumption.

Conjecture. The attractor in the neighbourhood of the prominent tangencies is the product of a continuum by a Cantor set.

Hence, near any image $x_n (n > 0)$ of the tangency point x_0, the attractor can be thought of as a collection of shifted parabolas (see figure 2), with a continuous mass distribution along the parabolas themselves. The latter ones expand as $\exp[\lambda_1(n, x_0)n]$ along the local y axis and contract according to $\lambda_2(n, x_0)$ along the local x axis. Therefore, the unstable manifold near x_n is approximately described by

$$y - y_n \sim \exp[(\lambda_1 - 2\lambda_2)n](x - x_n)^2. \tag{8}$$

Let us now consider a covering where the box closest to x_n is a square of size ε in the position indicated in figure 2. The asymptotic weight $p(n, \varepsilon)$ of this box, for $n \to \infty$ and $\varepsilon \to 0$, depends on whether ε is larger or smaller than R_n, where $R_n \simeq \exp[(2\lambda_2 - \lambda_1)n]$ is the radius of curvature at x_n. With the help of the conjecture, and calling α' the dimension measured in a box of axes

$$\delta_x \sim \varepsilon \exp(-n\lambda_2) \qquad \delta_y \sim \varepsilon \exp(-n\lambda_1) \tag{9}$$

around x_0, we obtain

$$p(n, \varepsilon) \simeq \begin{cases} \varepsilon^{\alpha'} \exp\{-[(\alpha' - 1)\lambda_1 + \lambda_2]n\} & \text{for} \quad \varepsilon < R_n \qquad (10a) \\ \varepsilon^{\alpha' - \frac{1}{2}} \exp[-(\alpha' - \frac{1}{2})\lambda_1 n] & \text{for} \quad \varepsilon > R_n. \qquad (10b) \end{cases}$$

Here, $(10a)$ is simply given by $\delta_x \delta_y^{(\alpha'-1)}$, since almost every parabola, intersecting the box along the vertical edges, yields a contribution δ_x to the mass. For $\varepsilon > R_n$ instead, the intersections occur mostly on the upper edge and the average width becomes of the order of $\sqrt{\delta_y}$ rather than δ_x, thus yielding $(10b)$. As shown by Grassberger *et al* (1987), α' depends on the effective Lyapunov exponents during the iterations leading to x_0. More precisely, it is given by the Kaplan–Yorke-type equation $\alpha' = 1 - \lambda_1'/\lambda_2'$, where $\lambda_1' = \lambda_1(m, x_{-m}) > 0$ is the Lyapunov exponent measuring the expansion along m pre-images of the x axis and $\lambda_2' < 0$ measures the contraction along the pre-images of the y axis (notice that expanding and contracting directions are interchanged when a trajectory passes through a primary tangency). The number m of pre-images is such that the length of the box was of the order of unity at time $-m$, i.e.

$$\varepsilon \exp(-n\lambda_1 - m\lambda_2') \sim 1. \tag{11}$$

Inserting (9) into (1), with $q > 1$, we notice that the influence of the nth image of a prominent tangency in the sum (1) is different, in the asymptotic limit, for $n > n(\varepsilon)$ or $n < n(\varepsilon)$, with

$$n(\varepsilon) = \frac{\ln \varepsilon}{2\lambda_2 - \lambda_1}. \tag{12}$$

The dominant contribution is found to be given by the box at the $n(\varepsilon)$th image (corresponding just to the case $\varepsilon = R_n$, shown in figure 2), and scales as

$$p[n(\varepsilon), \varepsilon]^q / \varepsilon^{\tau(q)} \sim \varepsilon^{q(2\alpha'-1)\alpha_L/(1+\alpha_L) - \tau(q)} \tag{13}$$

where we have introduced

$$\alpha_L = 1 - \lambda_1/\lambda_2 \tag{14}$$

as the effective 'Lyapunov' dimension of point x_n. Notice that α_L depends on the orbit leading from x_0 to x_n and does not coincide with α', which depends on the iterations leading from x_{-m} to x_0. From (11) and (12), we obtain

$$m = \frac{2(\lambda_1 - \lambda_2)}{\lambda_2'} n. \tag{15}$$

Taking then the limit $q \to \infty$, in (13), we see that each family $\{x_n | n > 0\}$ of tangencies yields a contribution $\varepsilon^{q\alpha - \tau(q)}$, with

$$\alpha \equiv \frac{(2\alpha' - 1)\alpha_L}{1 + \alpha_L} \geqslant D(\infty). \tag{16}$$

Notice that, although α can also be written as

$$\alpha = \lim_{\varepsilon \to 0} \frac{\ln p[n(\varepsilon), \varepsilon]}{\ln \varepsilon} \tag{17}$$

it cannot be considered as a pointwise dimension, since the ε boxes used in this limit are not centred around one single point. As a consequence, the corresponding $f(\alpha)$ cannot be interpreted as the Hausdorff dimension of any set of points. For a given ε, the set of heaviest boxes forms a 'volatile fractal' in the terminology of Herrmann and Stanley (1984). Also, α cannot be written as a sum of two partial dimensions, as is always the case for hyperbolic attractors. In the high-dissipation limit (α', $\alpha_L \to 1$), (16) yields $\alpha = \frac{1}{2}$, in agreement with the well known results for the logistic map (Grassberger *et al* 1987, Ott *et al* 1984). (There, this is indeed the pointwise dimension of a set of points, namely of the images of the critical point.) On the other hand, for conservative maps, where $\alpha' = \alpha_L = 2$, we obtain $\alpha = 2$, as it should.

In the intermediate cases, precise estimates require an assumption about the joint fluctuations of α' and α_L. In order to avoid such a difficulty, we limit ourselves to derive bounds on $\alpha_{\min}$. By substituting in (16) the average (most probable) value for both α' and α_L, an upper bound to $\alpha_{\min}$ is obtained. For the usual parameters $a = 1.4$ and $b = 0.3$, the result is

$$\alpha_{\min} \leqslant \alpha^{+}_{\min} = \frac{\alpha(1)[2\alpha(1) - 1]}{1 + \alpha(1)} \simeq 0.85 \tag{18}$$

where $\alpha(1)$ indicates the information dimension (2). In order to test this, and to obtain eventually a more precise numerical estimate, we computed $\alpha_{\min}$ by using a modified correlation method. We first located 100 prominent tangency points to an accuracy of better than 10^{-12} by following the unstable manifold of the fixed point and looking for points on it, where the next thirteen iterations led to a contraction of the unstable tangency vector by a factor $\leqslant 10^{-9}$. Then, we counted the number of points of a random trajectory (of length 2.6×10^8) which fell closer than ε to any of the first iterates of these prominent tangencies. This was made feasible by using a mesh in a way similar to that described by Theiler (1987). The result is shown on a log-log scale in figure 3. The bold curve represents the maximal number of points in an ε neighbourhood

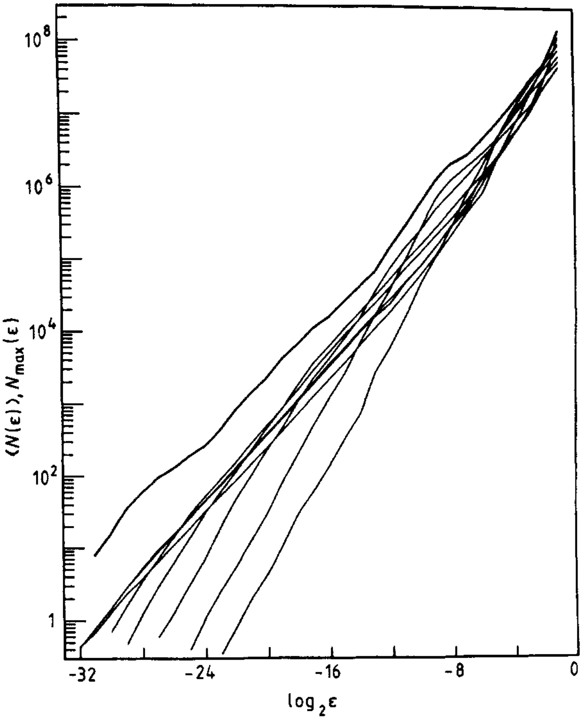

Figure 3. Correlation exponent near 100 prominent homoclinic tangencies and their images. Bold curve: maximal number of points in an ε neighbourhood of any image of any primary tangency. The slope yields α_{min}. Thin curves: average numbers of points in the ε neighbourhood of the nth images of primary tangencies ($0 \leq n \leq 6$).

of any of the 100 tangencies. From its slope, we estimate

$$\alpha_{min} \sim 0.76. \tag{19}$$

This value is in reasonable agreement with the box-counting estimate of Grassberger *et al* (1987). Since it is larger than α^-_{min} (given by (7)), we conclude that $f(\alpha)$ is not a straight line all the way down to α_{min}. The thin curves show, for each $n \in [0, 6]$, the average number of points in neighbourhoods of the nth iterates of the prominent tangencies. From these lines, we see that our basic results are indeed correct: for each ε there is a different iteration number $n(\varepsilon)$ which contributes maximally. For $n < n(\varepsilon)$, the slopes of all curves are roughly equal to the global dimension $D \approx 1.25$–1.3. For $n > n(\varepsilon)$, we observe another common slope smaller than α_{min}, as predicted in (9).

Let us now compare our results with two other recent attempts to compute α_{min} (Gunaratne and Procaccia 1987, Jensen 1987). Jensen counted points which fall into ε neighbourhoods of the seventh iterate of the prominent tangency nearest to the fixed point ('nearest' along the unstable manifold). This indeed gives a rough estimate of α_{min}, except for two sources of errors: firstly, by taking only one n value, with $n > n(\varepsilon)$ for all ε, one underestimates α_{min}. Secondly, by taking only the tangency closest to the fixed point, one overestimates α_{min} since the pointwise dimension α' is there equal to that at the fixed point (Grebogi *et al* 1987), and this is larger than the average. Numerically, these errors cancel exactly and Jensen's numerical result agrees with ours.

In contrast to this, the results of Gunaratne and Procaccia (1987) seem suspicious since they do not guarantee the correct $\alpha_{\min}$ in the two limits of conservative and of strongly dissipative systems.

The evaluation of $f(\alpha)$ for $\alpha < \alpha_c$ is performed by actually evaluating the sum in (1), or by counting the number $N(\varepsilon, \alpha)$ of boxes with effective local dimension α. Again, we consider only the dominant boxes at the nth iterates of prominent tangencies, with $n = n(\varepsilon)$ (equation (12)). The nth pre-images of these form a covering of the prominent tangencies with rectangles of height $\delta_y = \varepsilon \exp(-\lambda_1 n)$. Since the latter lie on a curve of length of order 1 transverse to the unstable manifold, the number of rectangles needed for the covering scales as $\delta_y^{1-f(\alpha')}$ and we obtain

$$N(\varepsilon, \alpha) \sim \varepsilon^{-f(\alpha)} \geqslant [\varepsilon \exp(-\lambda_1 n)]^{1-f(\alpha')}. \tag{20}$$

The final evaluation of $f(\alpha)$ is again not easy due to fluctuations in the Lyapunov exponents. A lower bound to $f(\alpha)$ is obtained by recalling (12) and inserting average values for both α' and α_L (with $f(\alpha') \approx \alpha(1)$):

$$f(\alpha^+_{\min}) \geqslant f^-(\alpha^+_{\min}) \equiv \frac{2[\alpha(1)-1]\alpha(1)}{1+\alpha(1)}. \tag{21}$$

Numerically, this gives 0.29 for the standard parameter values.

A further improvement to the lower bound (21) requires a conjecture on the joint distribution of α' and α_L on the set of boxes centred around the prominent tangencies. This is a very delicate question and we have not found any argument to justify a particular choice.

In conclusion, the approach described in the present letter has allowed to clarify the mechanism leading, in non-hyperbolic systems, to the creation of heavy boxes (small dimensions).

References

Badii R and Politi A 1987 *Phys. Rev.* A **35** 1288

Benzi R, Paladin G, Parisi G and Vulpiani A 1985 *J. Phys. A: Math. Gen.* **18** 2157

Bohr T and Rand D 1987 *Physica* **25D** 387

Collet P, Leibowitz J and Porzio A 1987 *J. Stat. Phys.* **47** 609

Frisch U 1984 *Proc. Int. School on Turbulence and Predictability in Geophysical Fluid Dynamics and Climate Dynamics* ed M Ghil (Amsterdam: North-Holland)

Fujisaka H 1983 *Prog. Theor. Phys.* **70** 1264

Grassberger P 1983 *Phys. Lett.* **97A** 227

—— 1984 *Proc. Conf. on Chaos in Astrophysics, Palm Coast, FL* ed J Perdang *et al* (Dordrecht: Reidel)

—— 1985 *Phys. Lett.* **107A** 101

Grassberger P, Badii R and Politi A 1987 *J. Stat. Phys.* to be published

Grassberger P and Procaccia I 1984 *Physica* **13D** 34

Grebogi C, Ott E and Yorke J A 1987 *Preprint* University of Maryland

Gunaratne G H and Procaccia I 1987 *Phys. Rev. Lett.* **59** 1377

Halsey T C, Jensen M H, Kadanoff L P, Procaccia I and Shraiman B 1986 *Phys. Rev.* A **33** 1141

Hentschel H G and Procaccia I 1983 *Physica* **8D** 435

Herrmann H J and Stanley H E 1984 *Phys. Rev. Lett.* **53** 1121

Jensen M H 1987 *Comment on 'Organization of Chaos'. Preprint* Nordita

Mori H *et al* 1987a *Prog. Theor. Phys.* **78** 511, 721

—— 1987b *Preprint* Kyushu University

Ott E, Withers W and Yorke J A 1984 *J. Stat. Phys.* **36** 687

Renyi A 1970 *Probability Theory* (Amsterdam: North-Holland)

Theiler J 1987 *Phys. Rev.* A **36** 4456